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ON SYMMETRIES IN OPTIMAL CONTROL

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Abstract Based on [5] we discuss the use of symmetries in solving optimal control problems. In particular a procedure for obtaining symmetries is given which can be performed **before** the actual calculation of the optimal control and optimal Hamiltonian.

1. Introduction

Consider an (unrestricted and smooth) Bolza problem of minimizing (w.r.t. $u(\cdot)$) the cost functional

$$J(x_0,u(\cdot)) = K(x(T)) + \int_0^T L(x(t),u(t))dt \tag{1.1}$$

under the dynamical constraints

$$\dot{x}(t) = f(x(t),u(t)),\ x(0) = x_0,\ x \in X,\ u \in U. \tag{1.2}$$

Here it is assumed that all data are smooth and that U equals $\mathbb{R}^m$ or an **open** subset of $\mathbb{R}^m$ (or more generally a manifold **without** boundary). For simplicity we also take X to be (an open subset of) $\mathbb{R}^n$.
In order to solve this optimal control problem the Maximum Principle tells us to introduce the pseudo-Hamiltonian $H: X \times \mathbb{R}^n \times U \to \mathbb{R}$ defined as

$$H(x,p,u) = p^T f(x,u) - L(x,u) \tag{1.3}$$

(with $p \in \mathbb{R}^n$ the co-state) and to consider the following set of differential equations

$$\text{(1.4a)}\quad \dot{x}_i = \frac{\partial H}{\partial p_i}(x,p,u) = f_i(x,u)$$
$$i = 1,\dots,n$$
$$\text{(1.4b)}\quad \dot{p}_i = -\frac{\partial H}{\partial x_i}(x,p,u)$$

with the (mixed) boundary conditions

$$\text{(1.5a)}\quad x(0) = x_0$$

$$\text{(1.5b)}\quad p(T) = -\frac{\partial K}{\partial x}(x(t))$$

where $x(T)$ is the solution of (1.4a) at time T for $x(0) = x_0$. A necessary condition for a control function $u^*(\cdot)$ on $[0,T]$ to be optimal (in the sense of (1.1)) is that for every $t \in [0,T]$

$$H(x^*(t),p^*(t),u^*(t)) = \max_{u \in U} H(x^*(t),p^*(t),u) \tag{1.6}$$

where $(x^*(\cdot),p^*(\cdot))$ is the solution of (1.4) with $u(\cdot) = u^*(\cdot)$ and boundary conditions (1.5). So the Maximum Principle leads to the following static optimization problem: Find for every $(x,p) \in X \times \mathbb{R}^n$ a $u^* \in U$ such that

$$H(x,p,u^*) = \max_{u \in U} H(x,p,u) \tag{1.7}$$

For simplicity we will throughout assume that the optimization problem (1.7) has a **unique smooth** solution $u^* = u^*(x,p)$, resulting in the **optimal Hamiltonian**

$$H^*(x,p) = H(x,p,u^*(x,p)) \tag{1.8}$$

Then the **optimal trajectory** $x^*(\cdot)$ is given by the solution of the differential equations

$$\text{(1.9)}\quad \begin{aligned} \dot{x}^* &= \frac{\partial H^*}{\partial p}(x^*,p^*) \\ \dot{p}^* &= -\frac{\partial H^*}{\partial x}(x^*,p^*) \end{aligned} \qquad x^*(0) = x_0,\ p^*(T) = -\frac{\partial K}{\partial x}(x^*(T))$$

and the **optimal control** in open loop form equals

$$u^*(t) = u^*(x^*(t),p^*(t)) \tag{1.10}$$

Equations (1.9) form a 2n-dimensional set of Hamiltonian differential equations, the solution of which may not be easy to obtain. On the other hand we know from **mechanics** that a basic tool in solving (or at least simplifying the solution of) Hamiltonian equations is to look for (a group of) **symmetries**. By Noether's theorem there corresponds to every (infinitesimal) symmetry a first **integral** of the differential equations, and the existence of k independent first integrals (in involution) reduces the 2n-dimensional set of equations to a (2n-2k)-dimensional set of Hamiltonian equations. The only difference with the usual case considered in mechanics is that in optimal control problems we usually have a **mixed** set of boundary conditions (cf. [4]).
Let us introduce some notation. For a smooth function $F: X \times \mathbb{R}^n \to \mathbb{R}$ its Hamiltonian vectorfield

$$\dot{x}_i = \frac{\partial F}{\partial p_i}(x,p)$$

(1.11) $$i = 1,\dots,n$$

$$\dot{p}_i = -\frac{\partial F}{\partial x_i}(x,p)$$

is denoted as X_F. The **Poisson bracket** of two functions F and G on $X \times \mathbb{R}^n$ is defined as

(1.12) $$\{F,G\} = X_F(G)$$

so that in coordinates

(1.13) $$\{F,G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial x_i} - \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial p_i}\right)$$

An (infinitesimal) **symmetry** for a Hamiltonian vectorfield X_H, or equivalently, for the Hamiltonian H, is another Hamiltonian vectorfield X_F satisfying

(1.14) $$X_F(H) = 0$$

Then by the properties of the Poisson bracket

(1.15) $$X_H(F) = \{H,F\} = -\{F,H\} = -X_F(H) = 0$$

and so F is a **first integral** for X_H. Conversely if F is a first integral for X_H, then X_F is a symmetry for X_H (or H).

The knowledge of symmetries for the optimal Hamiltonian H^* also simplifies the construction of the optimal control in **feedback form** $u^*(x^*(t),t) = u^*(t),p^*(t))$. For instance in the linear case (with $f(x,u) = Ax + Bu$, $L(x,u) = \frac{1}{2}x^TQx + \frac{1}{2}u^TRu$) symmetries for the optimal Hamiltonian vectorfield

(1.16) $$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & BR^{-1}B^T \\ Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

are in one-to-one correspondence with symmetries of the associated Riccati equation

(1.17) $$\dot{K} = -A^TK - KA - KBR^{-1}B^TK + Q$$

as noted in [11]. More generally in the nonlinear case symmetries for H^* yield symmetries for the Hamilton-Jacobi-Bellman equation

(1.18) $$\frac{\partial S}{\partial t}(x,t) = -H^*\left(x, \frac{\partial S}{\partial x}(x,t)\right),\quad S(x,T) = -K(x)$$

where $-S(x,t)$ is the Bellman value function (cf. [5]).

2. Symmetries for H^* obtained from the Hamiltonian system

As we saw in the introduction, the knowledge of symmetries for the optimal Hamiltonian H^* is useful for obtaining the optimal trajectory as well as for calculating the optimal control. Next question is of course: How to obtain these symmetries? The basic idea of our work [5,7,9], on which this paper is based, is to obtain symmetries for H^* directly from the initial data $f(x,u)$ and $L(x,u)$, **without** the necessity of first explicitly solving for $u^*(x,p)$ and constructing H^* (see also [4]). This is done as follows. Since we assumed U to be (an open part of) $\mathbb{R}^m$ equation (1.8) implies the first order conditions

(2.1) $$\frac{\partial H}{\partial u_j}\left(x,p,u^*(x,p)\right) = 0 \quad j = 1,\dots,m$$

Now notice that the equations

$$\dot{x}_i = \frac{\partial H}{\partial p_i}(x,p,u)$$

$$i = 1,\dots,n$$

(2.2) $$\dot{p}_i = -\frac{\partial H}{\partial x_i}(x,p,u)$$

$$y_j = \frac{\partial H}{\partial u_j}(x,p,u) \qquad j = 1,\dots,m$$

constitute a **Hamiltonian input-output system** ([7,9,5]) with inputs u and outputs y, and that the first-order condition for a control $u(\cdot)$ to be optimal is that the outputs y of this system resulting from this $u(\cdot)$ and the boundary conditions (1.5) are identically zero.

Let us now define symmetries of Hamiltonian **systems** ([6,7,9]). For a function F^e on $U \times \mathbb{R}^m$ let us define the Hamiltonian vector field X_{F^e} on $U \times \mathbb{R}^m$ as

$$\dot{u}_j = \frac{\partial F^e}{\partial y_j}(u,y)$$

(2.3) $$j = 1,\dots,m$$

$$\dot{y}_j = -\frac{\partial F^e}{\partial u_j}(u,y)$$

An (infinitesimal) **symmetry** for a Hamiltonian system (2.2) is defined as a **pair** of Hamiltonian vectorfields (X_F, X_{F^e}), with X_F a Hamiltonian vectorfield on $X \times \mathbb{R}^n$ and X_{F^e} a Hamiltonian vectorfield on $U \times \mathbb{R}^m$, such that

(2.4) $$X_F\left(H(x,p,u)\right) = -F^e(u,y)$$

where $y = \frac{\partial H}{\partial u}(x,p,u)$. (This implies that the flow of the vectorfield X_{F^e} leaves the external (input-output) behavior of the system (2.2) **invariant**, cf. [6,7].) Using the anti-symmetry of the Poisson bracket we obtain from (2.4)

(2.5) $$X_{H(x,p,u)}(F) = \{H(x,p,u),F(x,p)\} = F^e\left(u,\frac{\partial H}{\partial u}(x,p,u)\right)$$

Noting that $X_{H(x,p,u)}(F)$ equals the time-derivative $\frac{dF}{dt}$ of F **along** the Hamiltonian system (2.2), equations (2.5) express the fact that $\frac{dF}{dt}$ **only** depends on the inputs and outputs of the system. Therefore the pair (F,F^e) is called a **conservation law**. In mechanics usually the function F^e is such that $F^e(u,y) = 0$ for $u = 0$ (if the external force is zero then F is a conserved quantity). In our case however, we need the "dual" property $F^e(u,y) = 0$ for $y = 0$:

Theorem 1 [5,7] Let (F,F^e) be a conservation law for the Hamiltonian system (2.2) such that $F^e(u,0) = 0$ for all $u \in U$. Then F^e is a first integral for the optimal Hamiltonian, i.e.

$$(2.6) \qquad \{H^*(x,p),\ F(x,p)\} = 0$$

Concluding, first integrals for H^* can be obtained by looking for pairs (F,F^e) satisfying (2.5) and $F^e(u,0) = 0$ for all $u \in U$. Furthermore we obtain the interesting "extra bonus"

Theorem 2 [5] Let (F,F^e) be a conservation law as in Theorem 1. Denote the components of $u^*(x,p)$ by $u_j^*(x,p)$, $j = 1,\dots,m$. Then

$$(2.7) \qquad \begin{aligned} &\sum_{j=1}^{m} \frac{\partial^2 H}{\partial u_i \partial u_j}(x,p,u^*)\ \{F,u_j^*\} = \\ &= \sum_{j=1}^{m} \frac{\partial^2 H}{\partial u_i \partial u_j}(x,p,u^*)\ \frac{\partial F^e}{\partial y_j}(u^*,0) \end{aligned} \qquad i = 1,\dots,m$$

In particular, if u^* is a regular optimal control (i.e. the matrix $\frac{\partial^2 H}{\partial u_i \partial u_j}$ has full rank), then it follows from (2.7) that the components of u^* have to satisfy a set of first order partial differential equations

$$(2.8) \qquad \{F(x,p),u_j^*(x,p)\} = G_j\big(u^*(x,p)\big) \qquad j = 1,\dots,m$$

with $G_j\big(u^*(x,p)\big) = \frac{\partial F^e}{\partial y_j}\big(u^*(x,p),0\big)$. .

A special, but important class of conservation laws (F,F^e) as above are those with $F(x,p)$ of the form $p^T g(x)$ for an n-vector $g(x)$, and F^e **identically** zero. In this case (2.5) reduces to

$$(2.9) \qquad \{p^T f(x,u) - L(x,u),p^T g(x)\} = 0$$

Furthermore since $\{p^T f(x,u),p^T g(x)\} = p^T[f(x,u),g(x)]$, with [,] the Lie bracket, we obtain

$$(2.10a) \quad [g(x),f(x,u)] = 0$$

$$\text{for all } u \in U$$

$$(2.10b) \quad g\big(L(x,u)\big) = 0$$

This class of symmetries was studied in [4].

Example [5] Consider a mathematical pendulum in space ($\mathbb{R}^3$) with mass 1 and length 1. Suppose there is a horizontal field by which one can exert a force u_1 in the x-direction and a force u_2 in the y-direction. In spherical coordinates the dynamical equations are

$$(2.11) \qquad \begin{aligned} \ddot\theta &= -u_1 \sin\theta + u_2 \cos\theta \\ \ddot\phi &= -g \sin\phi + u_1 \cos\theta \cos\phi + u_2 \sin\theta \cos\phi \end{aligned}$$

with $(\phi,\theta) \in S^2$ (the unit sphere). Therefore the state space is $X = TS^2$, with local coordinates

$$x_1 = \phi_1, x_2 = \theta,\ x_3 = \dot\phi,\ x_4 = \dot\theta.$$

Let us take $L(x,u) = \frac{1}{2}(u_1^2 + u_2^2)$. . An evident candidate for symmetry vectorfield on X is $g(x) = \frac{\partial}{\partial x_2}$, with corresponding Hamiltonian p_2. Calculating

$$\{p^T f(x,u) - L(x,u),p_2\} =$$

$$= \{p_1x_3+p_2x_4+p_3(-g\sin x_1+u_1\cos x_2\cos x_1+u_2\sin x_2\cos x_1) +$$

$$+ p_4(-u_1\sin x_2+u_2\cos x_2) - \tfrac{1}{2}(u_1^2+u_2^2),p_2\} =$$

$$= u_1p_3\sin x_2\cos x_1 - u_2p_3\cos x_2\cos x_1 +$$

$$+ u_1p_4\cos x_2 + u_2p_4\sin x_2.$$

Furthermore

$$y_1 = \frac{\partial H}{\partial u_1} = p_3 \cos x_2\cos x_1 - p_4\sin x_2 - u_1$$

$$y_2 = \frac{\partial H}{\partial u_2} = p_3 \sin x_2\cos x_1 - p_4\cos x_2 - u_2$$

and hence

$$\{p^T f(x,u) - L(x,u),\ p_2\} =$$

$$= u_1(y_2+u_2) - u_2(y_1+u_1) = u_1y_2 - u_2y_1$$

So $(p_2,\ u_1y_2 - u_2y_1)$ is a conservation law satisfying the conditions of Theorems 1, 2. Therefore

$$(2.12a) \quad \{H^*(x,p),p_2\} = 0$$

$$(2.12b) \quad \{p_2,u_1^*\} = -u_2^* \quad ,\{p_2,u_2^*\} = u_1^*$$

In this case u_1^*, u_2^* and H^* can be immediately computed, thanks to the simple form of $L(x,u)$. However, suppose $L(x,u) = \frac{1}{4}(u_1^2+u_2^2)^2$. Then still $(p_2,u_1y_2 - u_2y_1)$ is a conservation law, as can be easily checked.
In general, if $L(x,u)$ is of the form

$$L(x,u) = h(x_1)\cdot k\big(\tfrac{1}{2}(u_1^2 + u_2^2)\big),$$

with h and k arbitrary smooth functions, then $(p_2,u_1y_2 - u_2y_1)$ is a conservation law. Hence by (2.12b) one knows a **priori** that

$$(2.13) \qquad \begin{aligned} \frac{\partial^2 u_1^*}{\partial x_2^2} &= \{p_2,\{p_2,u_1^*\}\} = -\{p_2,u_2^*\} = -u_1^* \\ \frac{\partial^2 u_2^*}{\partial x_2^2} &= \{p_2,\{p_2,u_2^*\}\} = \{p_2,u_1^*\} = -u_2^* \end{aligned}$$

Consequently in all these cases u_1^* and u_2^* have to be of the form

$$(2.14) \qquad a \sin x_2 + b \cos x_2$$

with a and b functions of x_1, x_3, x_4, p_1, p_2, p_3 and p_4.

3. Time-derivatives of optimal controls.

For every Hamiltonian vectorfield X_H there always exists one "trivial" symmetry or first integral. This is the vectorfield X_H respectively the Hamiltonian H itself, as immediately follows from the antisymmetry of the Poisson bracket

$$(3.1) \qquad \{H,H\} = 0$$

Equation (3.1) expresses the well-known "conservation of energy" for a Hamiltonian set of differential equations.

How can this simple fact be applied to our case? With regard to Theorem 1 we do not gain anything new; the optimal Hamiltonian H^* is a first integral for the optimal equations (1.9). However, with regard to Theorem 2 we obtain

Theorem 3 Let $H^*(x,p) = H(x,p,u^*(x,p))$ denote the optimal Hamiltonian. Then

$$(3.2) \qquad \sum_{j=1}^{m} \frac{\partial^2 H}{\partial u_i \partial u_j}(x,p,u)\{H^*(x,p),u_j^*(x,p)\} =$$

$$= \{\frac{\partial H}{\partial u_i}(x,p,u),\ H(x,p,u)\} \qquad i = 1,\ldots,m$$

where everything is evaluated in $u = u^*(x,p)$.

Proof Since $\frac{\partial H}{\partial u_j}(x,p,u^*(x,p)) = 0$, $j = 1,\ldots,m$, for any time t, we also have

$$(3.3) \qquad \{H^*(x,p),\ \frac{\partial H}{\partial u_j}(x,p,u^*(x,p))\} = 0 \qquad j = 1,\ldots,m$$

Writing out

$$0 = \{H^*(x,p),\ \frac{\partial H}{\partial u_j}(x,p,u^*(x,p))\} =$$

$$= \{H^*(x,p),\ \frac{\partial H}{\partial u_j}(x,p,u)\}_{u=u^*(x,p)} +$$

$$+ \sum_{k=1}^{m} \frac{\partial^2 H}{\partial u_k \partial u_j}\{H^*(x,p),\ u_k^*(x,p)\}$$

Moreover

$$\{H^*(x,p),\ \frac{\partial H}{\partial u_j}(x,p,u)\}_{u=u^*(x,p)} =$$

$$\{H(x,p,u),\ \frac{\partial H}{\partial u_j}(x,p,u)\}_{u=u^*(x,p)} +$$

$$+ \sum_{k=1}^{m} \frac{\partial H}{\partial u_k}(x,p,u^*(x,p))\cdot$$

$$\cdot\{u_k^*(x,p),\ \frac{\partial H}{\partial u_j}(x,p,u)\}_{u=u^*(x,p)} =$$

$$= \{H(x,p,u),\ \frac{\partial H}{\partial u_j}(x,p,u)\}_{u=u^*(x,p)} \qquad \square$$

Since $\{H^*(x,p),\ u_j^*(x,p)\}$ can be denoted as $\frac{du_j^*}{dt}$, with $\frac{d}{dt}$ differentiation along the optimal equations (1,9), we rewrite (3.3) in the form

$$(3.4) \qquad \sum_{j=1}^{m} \frac{\partial^2 H}{\partial u_i \partial u_j}\frac{du_j^*}{dt} = \{\frac{\partial H}{\partial u_i},\ H\} \qquad j = 1,\ldots,m$$

with everything evaluated in $(x,p,u^*(x,p))$. Especially when u^* is a regular optimal control this is a useful formula for the time-derivative of the optimal control. (Notice that in many cases the right-hand side of (3.4) can be determined without explicitly solving for u^* and H^*.) For instance in case

$$L(x,u) = \sum_{j=1}^{m} u_j^2, \quad f(x,u) = g_0(x) + \sum_{j=1}^{m} u_j g_j(x)$$

we obtain

$$(3.5) \qquad \frac{du_j^*}{dt} = \{p^T g_j(x) - u_j,\ p^T g_0(x) +$$

$$+ \sum_{k=1}^{m} u_k^* p^T g_k(x) - \sum_{k=1}^{m} u_k^2\}$$

$$= p^T[g_j,g_0] + \sum_{k=1}^{m} u_k^*[g_j,g_k]$$

From this we immediately obtain that, if all Lie brackets of $g_0, g_1, \ldots, g_m$ are zero, then u^* is constant. This fact was recognized for single input bilinear systems in [1].

4. Higher-order symmetries

It is known from physics that apart from considering symmetries for (Hamiltonian) equations, one may also look for symmetries of the **prolonged** Hamiltonian equations; these are called higher-order symmetries. As noted already in [8], see also [10], also Hamiltonian **systems** can be prolonged. For the first-order prolongation this is done by prolonging the pseudo-Hamiltonian $H(x,p,u)$ to the pseudo-Hamiltonian

$$(4.1) \qquad \dot{H}(x,p,\dot{x},\dot{p},u,\dot{u}) =$$

$$\sum_{i=1}^{m}\left(\frac{\partial H}{\partial x_i}\dot{x}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right) + \sum_{j=1}^{m} \frac{\partial H}{\partial u_j}\dot{u}_j$$

on the 4n-dimensional state space $(x,p,\dot{x},\dot{p})$ with Poisson bracket

$$(4.2) \qquad \{F(x,p,\dot{x},\dot{p}),\ G(x,p,\dot{x},\dot{p})\} =$$

$$\sum_{i=1}^{n}\left(\frac{\partial F}{\partial \dot{p}_i}\frac{\partial G}{\partial x_i} - \frac{\partial F}{\partial \dot{x}_i}\frac{\partial G}{\partial p_i} + \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial \dot{x}_i} - \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial \dot{p}_i}\right)$$

The inputs of the resulting prolonged Hamiltonian system are u and $\dot{u}$, while the outputs are given as

$$(4.3a) \qquad y_j = \frac{\partial \dot{H}}{\partial \dot{u}_j} = \frac{\partial H}{\partial u_j} \qquad j = 1,\ldots,m$$

(4.3b) $$\dot{y}_j = \frac{\partial \dot{H}}{\partial u_j} = \frac{\partial}{\partial u_j}\left(\sum_{i=1}^{n}\left(\frac{\partial H}{\partial u_i}\dot{x}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right) + \sum_{k=1}^{m}\frac{\partial H}{\partial u_k}\dot{u}_k\right)$$

Substitution of $\dot{x}_i = \frac{\partial H}{\partial p_j}$, $\dot{p}_i = -\frac{\partial H}{\partial x_i}$ into (4.3b) yields

(4.4) $$\dot{y}_j = \sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial u_j}\right)\frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i}\left(\frac{\partial H}{\partial u_j}\right)\frac{\partial H}{\partial x_i}\right) + \sum_{k=1}^{m}\frac{\partial}{\partial u_j}\left(\frac{\partial H}{\partial u_k}\right)\dot{u}_k = \frac{d}{dt}\left(\frac{\partial H}{\partial u_j}\right) = \frac{dy_j}{dt}$$

where $\frac{d}{dt}$ means differentiation along the Hamiltonian system (2.2).

A conservation law for the prolonged Hamiltonian system consists of a pair of functions $F(x,p,\dot{x},\dot{p})$ and $F^e(u,y,\dot{u},\dot{y})$ satisfying

(4.5) $$\{\dot{H}(x,p,\dot{x},\dot{p}),\ F(x,p,\dot{x},\dot{p})\} = F^e(u,y,\dot{u},\dot{y})$$

where y and $\dot{y}$ are given by (4.3). If moreover $F^e(u,0,\dot{u},0) = 0$ for all u and $\dot{u}$, then it follows from Theorem 1 that F is a first integral for the prolonged optimal Hamiltonian, i.e.

(4.6) $$\{\dot{H}^*(x,p,\dot{x},\dot{p}),\ F(x,p,\dot{x},\dot{p})\} = 0$$

It can be seen that if $(F(x,p),\ F^e(u,y))$ is a conservation law for the original Hamiltonian system (2.2) then $(\dot{F},\dot{F}^e)$ will be a conservation law for the prolonged system. Moreover if $F^e(u,0) = 0$ for all u then $\dot{F}(u,0,\dot{u},0) = 0$ for all u and $\dot{u}$, and so $\dot{F}$ will be a first integral for $\dot{H}^*$. However, the class of higher-order symmetries is much bigger than the symmetries obtained in this way. Especially in the singular optimal control case higher-order symmetries seem to be of use, for instance for a nondegenerate analog of Theorem 2. In this context we remark that in the singular case we can derive formulae similar to (3.2) by considering repeated Poisson brackets

(4.7) $$\{H^*(x,p),\{H^*(x,p),\ldots\{H^*(x,p),\frac{\partial H}{\partial u_j}(x,p,u^*(x,p))\}\ldots\}\} = 0$$

5. Symmetries and feedback

Let us consider again the optimal control problem (1.1). Assume that we first apply **feedback** $u = \alpha(x,v)$, $v \in \mathbb{R}^m$, with $\frac{\partial \alpha}{\partial v}$ non-singular. We obtain the transformed control system

(5.1) $$\dot{x} = \tilde{f}(x,v) := f(x,\alpha(x,v))$$

with running cost

(5.2) $$\tilde{L}(x,v) := L(x,\alpha(x,v))$$

resulting in the pseudo-Hamiltonian

(5.3) $$\tilde{H}(x,p,v) = p^T\tilde{f}(x,v) - \tilde{L}(x,v)$$

It is immediate that the two optimal controls u^*, respectively v^*, are linked by the formula

(5.4) $$u^* = \alpha(x^*,v^*)$$

and that $H^*(x,p) = \tilde{H}^*(x,p)$. However the Hamiltonian **systems** corresponding to H and $\tilde{H}$ are really different, and so are their symmetries. Hence in general by considering H we obtain **different** symmetries for $H^* = \tilde{H}^*$ than by considering $\tilde{H}$. This fact was used in [4] in order to obtain conservation laws (F,F^e) for H with F^e identically zero (and F of the form $p^Tg(x)$). For a discussion of the rather puzzling connection between feedback transformations and the resulting Hamiltonian systems we refer to [5].

6. Conclusion

We have argued that the existence of symmetries for the optimal Hamiltonian considerably simplifies the solution of optimal control problems. It would be very useful to determine the class of optimal control problems which give rise to **completely integrable** optimal Hamiltonians [2]. We have given a procedure to obtain symmetries for the optimal Hamiltonians which avoids the explicit calculation of the optimal control and Hamiltonian. These symmetries have the interpretation of being symmetries for the associated Hamiltonian systems. For other approaches for finding symmetries we refer to [9], see also [3].

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